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EXPLORING CREATION WITH
# ADVANCED PHYSICS
1st EDITION

Wile

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Introduction

There are probably no concepts more important in physics than the two listed in the title of this module. In your first-year physics course, I am sure that you learned quite a lot about both of these concepts. You certainly did not learn everything, however. Whether we are talking about units or vectors, there is simply too much information to possibly learn in just one year. As a result, we will take another look at both of these concepts in this first module. This will help you "warm up" to the task of recalling all of the things you learned in your first-year physics course, and it will help you to learn both of these valuable concepts at a much deeper level.

Units Revisited

Almost regardless of the physics course, units should be covered first, because a great deal of physics is based on properly analyzing units. In your first-year course, you were taught how to solve problems such as the one in the following example:

EXAMPLE 1.1

A sample of iron has a mass of 254.1 mg. How many kg is that?

In this problem, we are asked to convert from milligrams to kilograms. We cannot do this directly, because we have no relationship between mg and kg. However, we do know that a milligram is the same thing as 0.001 grams and that a kilogram is the same thing as 1,000 grams. Thus, we can convert mg into g, and then convert g into kilograms. To save space, we can do that all on one line:

$$\frac{254.1 \; \cancel{mg}}{1} \times \frac{0.001 \; g}{1 \; \cancel{mg}} \times \frac{1 \; kg}{1,000 \; g} = 0.0002541 \; kg = 2.541 \times 10^{-4} \; kg$$

The sample of iron has a mass of $\underline{2.541 \times 10^{-4} \; kg}$.

Did this example help dust the cobwebs out of your mind when it comes to units? It should all be review for you. I converted the units using the factor-label method. Because this is a conversion, I had to have the same number of significant figures as I had in the beginning, and even though it was not necessary, I reported the answer in scientific notation. If you are having trouble remembering these techniques, then go back to your first-year physics book and review them.

There are a couple of additional things I want you to learn about units. I am not going to show you any new techniques; I am just going to show you new ways of applying the techniques that you should already know. Consider, for example, the unit for speed. The standard unit for

speed is $\frac{m}{sec}$. However, *any* distance unit over *any* time unit is a legitimate unit for speed. Since that is the case, we should be able to convert from one unit for speed to any other unit for speed. Study the following example to see what I mean.

EXAMPLE 1.2

As of 2001, the record for the fastest lap at the Indianapolis 500 ("The greatest Spectacle in Racing") was held by Arie Luyendyk. He averaged a speed of 225.2 miles per hour over the entire 2.5-mile stretch of the Indianapolis speedway. What is that speed in meters per second?

This problem requires us to make two conversions. To get from miles per hour to meters per second, we must convert miles to meters. Then, we must convert hours to seconds. This is actually easy to do. Remember, in miles per hour, the unit "miles" is in the numerator of the fraction and the unit "hours" is in the denominator. Also remember that there are 1609 meters in a mile and that 1 hour is the same as 3600 seconds.

$$\frac{225.2 \text{ miles}}{1 \text{ hr}} \times \frac{1609 \text{ meters}}{1 \text{ mile}} \times \frac{1 \text{ hr}}{3600 \text{ sec}} = 100.7 \frac{\text{meters}}{\text{sec}}$$

Although there is nothing new here, you probably haven't seen a conversion done in this way. Despite the fact that the unit for speed is a derived unit, I can still do conversions on it. I could have just converted miles to meters and gotten the unit meters/hour. I also could have just converted hours to seconds and gotten miles/second. In this case, however, I did both. That way, I ended up with meters/second. When working with derived units, remember that you can convert any or all units that make up the derived unit. Thus, 225.2 miles per hour is the same thing as 100.7 meters per second. Please note that although 3600 has only 2 significant figures, the number is actually infinitely precise, because there are *exactly* 3600 seconds in an hour. Thus, it really has an infinite number of significant figures. This is why I say that the best rule of thumb is to always end your conversion with the same number of significant figures as that with which you started your conversion.

Okay, we are almost done reviewing units. There is just one more thing that you need to remember. Sometimes, units have exponents in them. You were probably taught how to deal with this fact in your first-year physics course, but we need to review it so that you *really* know how to deal with it.

EXAMPLE 1.3

One commonly used unit for volume is the cubic meter. After all, length is measured in meters, and volume is length times width times height. The more familiar unit, however, is

cubic centimeters (cc) which is often used in medicine. If a doctor administers 512 cc of medicine to a patient, how many cubic meters is that?

Once again, this is a simple conversion. If, however, you do not think as you go through it, you can mess yourself up. We need to convert cubic centimeters to cubic meters. Now remember, a cubic centimeter is just a cm^3 and a cubic meter is just a m^3. We have no relationship between these units, but we do know that 1 cm = 0.01 m. That's all we need to know, as long as we think about it. Right now, I have the following relationship:

$$1 \text{ cm } = 0.01 \text{ m}$$

This is an equation. I am allowed to do something to one side of the equation as long as I do the exact same thing to the other side of the equation. Okay, then, let's cube both sides of the equation:

$$(1 \text{ cm})^3 = (0.01 \text{ m})^3$$

$$1 \text{ cm}^3 = 0.000001 \text{ m}^3$$

Now look what we have. We have a relationship between cm^3 and m^3, exactly what we need to do our conversion!

$$\frac{512 \ \cancel{\text{cm}^3}}{1} \times \frac{0.000001 \text{ m}^3}{1 \ \cancel{\text{cm}^3}} = 5.12 \times 10^{-4} \text{ m}^3$$

So 512 cc's is the same as $\underline{5.12 \times 10^{-4} \text{ m}^3}$.

When most students do a conversion like the one in the example without thinking, they simply use the relationship between cm and m to do the conversion. That, of course does not work, because the cm^3 unit does not cancel out, and you certainly don't get the m^3 unit in the end:

$$\frac{512 \text{ cm}^3}{1} \times \frac{0.01 \text{ m}}{1 \ \cancel{\text{cm}}} = 5.12 \text{ m} \cdot \text{cm}^2$$

Do you see what happened? The cm unit canceled one of the cm out of cm^3, but that still left cm^2. Also, since m is the unit that survives from the conversion relationship, you get the weird unit of m·cm^2! When you are working with units that have exponents in them, you need to be very careful about how you convert them. At the risk of "beating this to death," I want to combine the previous two examples into one more example.

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EXAMPLE 1.4

The maximum acceleration of a certain car is 21,600 miles per hour^2. What is the acceleration in feet per second^2?

Once again, this is a derived unit, but that should not bother you. All we have to do is convert miles into feet and hours^2 into seconds^2. There are 5280 feet in a mile, so that conversion will be easy. We do not know a conversion between hours^2 and seconds^2, but we do know that:

$$1 \text{ hour} = 3600 \text{ seconds}$$

To get the conversion relationship between hours^2 and seconds^2, then, we just square both sides:

$$(1 \text{ hour})^2 = (3600 \text{ seconds})^2$$

$$1 \text{ hour}^2 = 1.296 \times 10^7 \text{ seconds}^2$$

Once again, please note that the conversion relationship between hours and seconds is exact. Thus, both numbers have an infinite number of significant figures. That's why I reported all digits when I squared 3600 seconds.

Now that I have the conversion relationships that I need, the conversion is a snap:

$$\frac{21,600 \text{ miles}}{1 \text{ hr}^2} \times \frac{5280 \text{ ft}}{1 \text{ mile}} \times \frac{1 \text{ hr}^2}{1.296 \times 10^7 \text{ sec}^2} = \underline{8.80 \frac{\text{ft}}{\text{sec}^2}}$$

Notice once again that had I not squared the conversion relationship between hours and seconds, the units would not have worked out. In order for hr^2 to cancel, the unit hr^2 had to be in the problem. That's why it is important to watch the units and make sure they cancel properly.

Make sure that you understand how to manipulate units this way by performing the following "on your own" problems.

ON YOUR OWN

1.1 The speed limit on many highways in the United States is 65 miles per hour. What is the speed limit in centimeters per second? (There are 1609 meters in a mile.)

1.2 The size of a house is 1600 square feet. What is the square yardage of the house?

1.3 The standard energy unit is the Joule ($\frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2}$). An alternative energy unit is the erg

($\frac{\text{g} \cdot \text{cm}^2}{\text{sec}^2}$). I am serious. It is called the erg! Convert 151 Joules into ergs.

A Review of Vectors

In your first-year physics course, you learned about **vectors**. A quantity is called a vector if it contains information about both magnitude and direction. A quantity is called a **scalar** if it contains only information about magnitude and no information about direction. For example, if I tell you that my car is moving at 55 miles per hour, I am giving you a scalar quantity. The speed of the car tells you "how much" (magnitude), but it does not tell you in what direction the car is moving. If I tell you that my car is moving at 55 miles per hour *due west*, then I am giving you a vector quantity. Not only do you know how fast I am traveling (the magnitude), but you also know which way I am heading (the direction). You should have already learned that we call the scalar quantity in this case **speed**, and we call the vector quantity **velocity**.

Since vectors contain information about both magnitude and direction, we use arrows to represent them. The length of the arrow represents the magnitude, while the direction in which the arrow points represents the direction. When I refer to vectors, I will emphasize that they are vectors by placing them in boldface type. For example, if you see "**A**" in an equation, you will know that it is a vector quantity because it is in boldface type. If you see "A" in an equation, you will know that it is a scalar quantity because it is not in boldface type.

Now although vectors and scalars are quite different, they can interact mathematically. For example, I can multiply a vector by a scalar. What happens if I do that? Well, a vector contains information about magnitude and direction, while a scalar contains information only about magnitude. If I multiply a vector by a scalar, then, the multiplication will affect the *magnitude* of the vector. If the scalar happens to be negative, it does affect the direction of the vector as well. Study the following figure to see what I mean.

FIGURE 1.1
Scalar Multiplication

A   2·A   ½ ·A   -2·A
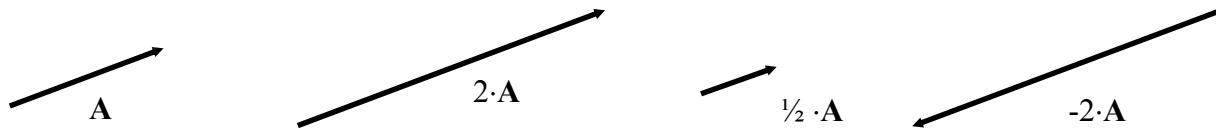
Notice in the figure that I start with an arrow which represents vector **A**. It has a magnitude (the length of the arrow) and a direction (the direction in which the arrow points). When I multiply by 2 (a scalar), what happens? The arrow points in the same direction, but it is twice as long. That's because when I multiply a vector by a positive scalar, I multiply its magnitude by the scalar, but I leave the direction alone. Thus, the length of the arrow (the magnitude) changes, but the direction does not. In the same way, when I multiply **A** by ½, the length of the arrow changes (it gets cut in half), but the direction does not.

Now look at the last arrow in the figure. This arrow represents what happens to the vector **A** when I multiply by a negative scalar. When you multiply a vector by a *positive* scalar, the direction of the vector does not change. However, when you multiply by a *negative* scalar, the direction of the vector becomes *opposite* of what it once was. Thus, when I multiply **A** by negative 2, the length of the vector increases by a factor of 2, but the vector also points in the opposite direction. Vector **A** points towards the upper right hand corner of the figure, while the vector -2·**A** is twice as long and points to the lower left-hand corner of the figure.

When vectors are multiplied by scalars, then, the result is another vector. As you already learned in your first-year physics course, vectors can not only be multiplied by scalars, but they can also be added to other vectors. The result in that case is a vector as well. To review how vectors are added to one another, however, we must first review the way that vectors can be mathematically represented.

A two-dimensional vector can be mathematically represented in one of two ways. It can be represented by its magnitude and an angle, or it can be written in terms of two perpendicular **components**. Although *any* two perpendicular components can be used to define a two-dimensional vector, we typically use *horizontal* and *vertical* components. These two ways of mathematically representing a vector, as well as the relationships between them, are summarized in Figure 1.2.

FIGURE 1.2
A Two-Dimensional Vector

vertical (y) axis

horizontal (x) axis

A

θ

vertical component (A_y)

horizontal component (A_x)

<u>Vector A can be written as</u>:

(1) A (magnitude) and θ

or

(2) A_x and A_y

<u>Mathematical Relationships</u>

$A_x = A \cdot \cos\theta$

$A_y = A \cdot \sin\theta$

$\tan\theta = \dfrac{A_y}{A_x}$

$A = \sqrt{A_x^2 + A_y^2}$

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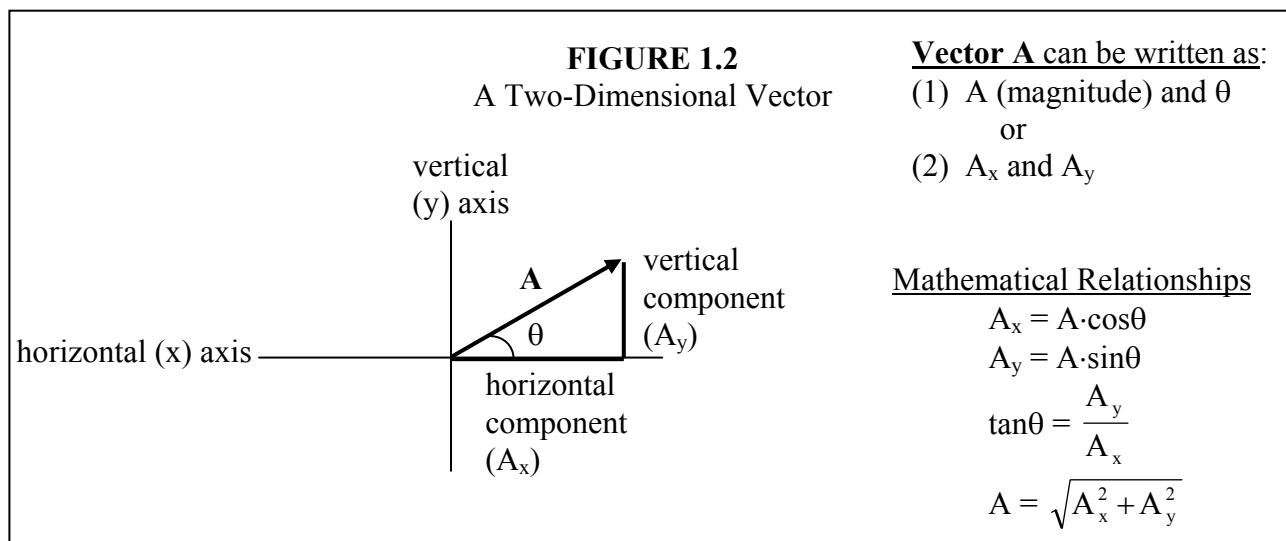
Notice, then, that the vector **A** can be represented with a magnitude (A) and a direction (θ). For example, if I told you that there was buried treasure 10.5 miles away from your current location at an angle of 50.4^o, I would be giving you a vector to describe the location of the treasure. I would be giving you that vector in terms of its magnitude and angle. Alternatively, I could tell you that to get to the treasure, you need to walk straight for 6.5 miles then turn left and walk another 8.1 miles. That is also a vector which represents the location of the treasure, but the vector is given in terms of two components rather than a magnitude and direction. Of course, since both methods represent the same vector, they should be related to one another. The figure summarizes the various means by which the components of a vector relate to the vector's magnitude and direction.

Now we can move on to adding vectors. In order to graphically add vectors, we simply put the tail of the second vector at the head of the first vector, and then we draw an arrow from the tail of the first to the head of the second. The new arrow is the graphical representation of the final vector, which is the sum of the two original vectors. Alternative, we can mathematically add vectors. To do this, we simply add their x-components to get the final vector's x-component, and we add their y-components to get the final vector's y-component. These processes are summarized in Figure 1.3.

FIGURE 1.3
Adding Vectors

Two original vectors - **A** and **B**:

Adding the vectors graphically:

$$C = A + B$$

$$C = B + A$$

Adding the vectors mathematially:

$$C_x = A_x + B_x$$

$$C_y = A_y + B_y$$
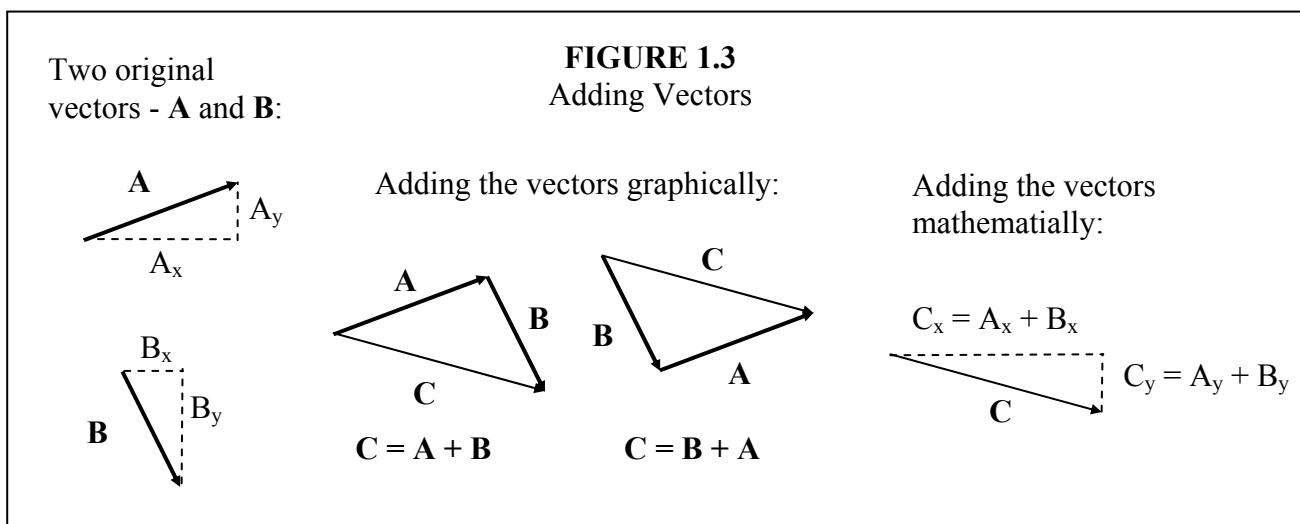
On the left-hand side of the figure, two vectors (**A** and **B**) are drawn. Their horizontal (x) components (A_x and B_x) as well as their vertical (y) components (A_y and B_y) are shown. In the middle of the figure, the graphical method for adding vectors is shown. To add vectors **A** and **B**, we put the tail of **B** on the head of **A**. Then, we draw an arrow from the tail of **A** to the head of **B**. The resulting arrow is the sum of **A** + **B**. Notice that the figure also shows the graphical representation of **B** + **A**. At first glance, you might think that the result of **A** + **B** is different than the result of **B** + **A**. That's not true. Remember, a vector is determined by its magnitude and direction. Both the result of **A** + **B** and the result of **B** + **A** have the same magnitude (length) and direction (they both point in the same direction). Thus, they each represent the same vector. They are simply shifted relative to one another.

That brings up an important thing to remember about vectors. They can be moved around without affecting their value. As long as you do not change the length of the arrow or the direction in which the arrow points, you can move it all over the place without changing the meaning of the vector at all.

Since physics is inherently a problem-solving science, the graphical approach to adding vectors is not enough. It gives us a visual picture of the sum of vectors, but it does not give us any numbers with which to work. Thus, we will mostly use the mathematical method for adding vectors, which is shown in the right hand portion of the figure. To add two vectors mathematically, we simply add the horizontal components together. This gives us the horizontal component of the final vector. We then add the vertical components together, and that gives us the vertical component of the vector. In the figure, then, the x-component of the final vector is simply $A_x + B_x$, while the y-component is simply $A_y + B_y$. Let me go through a quick example problem to jog your memory on all of this.

EXAMPLE 1.5

An explorer is given directions on how to get to a particular place. He is told to travel for 15.2 miles at a heading of 30.0 degrees and then to turn and travel 30.4 miles at a heading of 170.2 degrees. What is the vector which describes the explorer's final position? Solve this problem both graphically and mathematically.

Let's start with the graphical method. First, we have to figure out how to draw the first vector. Well, the magnitude of the first vector (let's call it **A**) is 15.2 miles. The angle is 30.0 degrees. We can represent that vector as:
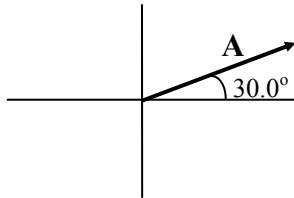
Please realize that the drawing simply approximates the values of the vector to give us a visual idea of what it looks like. We know that if the arrow pointed straight along the horizontal axis to the right, the angle would be 0 degrees. If it pointed up along the vertical axis, the angle would be 90.0 degrees. If it bisected those two, it would be pointing at 45 degrees. An angle of 30.0 degrees, then, is close to bisecting the two axes, but not quite.

Next, we put the tail of the second vector at the head of the first. The second vector has a magnitude of 30.4 miles and an angle of 170.2 degrees. Putting it on the head of the first vector gives us the following drawing:
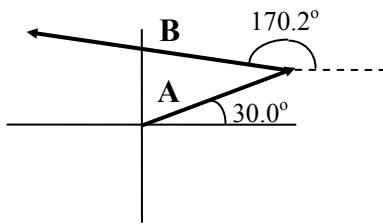
Remember, if the vector pointed straight up, its angle would be 90.0 degrees. If it pointed directly to the left, its angle would be 180.0 degrees. Thus, an angle of 170.2 degrees points almost directly to the left, but just a little up. In addition, the magnitude is twice that of the first vector, so the arrow is twice as long as the first.

Adding the vectors is now a snap. We just draw an arrow from the tail of the first to the head of the second.
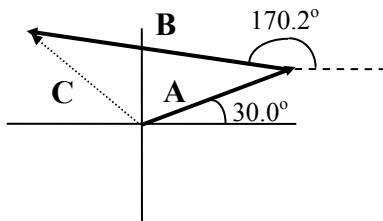
The arrow labeled **C** represents the sum of the two vectors.

Now, although that gives us a picture of the vector which represents the explorer's final position, it is only a picture. To get numbers which describe this vector, we must add the two vectors mathematically. To do that, we must get the horizontal and vertical components of each vector and add them together. Figure 1.2 gives the relationships between the components of a vector and its magnitude and direction, so that's not too bad:

$$A_x = (15.2 \text{ miles}) \cdot \cos(30.0) = 13.2 \text{ miles}$$

$$A_y = (15.2 \text{ miles}) \cdot \sin(30.0) = 7.60 \text{ miles}$$

$$B_x = (30.4 \text{ miles}) \cdot \cos(170.2) = -30.0 \text{ miles}$$

$$B_y = (30.4 \text{ miles}) \cdot \sin(170.2) = 5.17 \text{ miles}$$

Notice that I used the rules of significant figures here. If you have forgotten those rules, go back to your first-year physics course (or chemistry if you took that course) and review significant figures so that you understand why I rounded the answers where I did.

To get the x-component of the sum (vector **C**), we just add the two x-components together, and to get the y-component of the sum, we just add the two y-components together.

$$C_x = A_x + B_x = 13.2 \text{ miles} + -30.0 \text{ miles} = -16.8 \text{ miles}$$

$$C_y = A_y + B_y = 7.60 \text{ miles} + 5.17 \text{ miles} = 12.77 \text{ miles}$$

In this case, I had to add numbers, and the significant figure rules are different for addition and subtraction as compared to multiplication and division. Make sure you remember the difference!

The answer, then, is that <u>the explorer's position has an x-component of -16.8 miles and a y-component of 12.77 miles</u>. This means the explorer is 16.8 miles left (west) of his starting position and 12.77 miles up (north) from his starting position.

Before I leave this review, there is one more important thing of which I must remind you. When dealing with the angle associated with a vector, you need to define the angle *counterclockwise from the positive x-axis*, as shown in the example above. If you define the angle in that way, the mathematics will always work. Thus, if you find yourself working with a vector whose angle is *not* defined in that way, change the angle so that it is defined properly.

If you have the components of a vector and need to get its magnitude and its angle, you can use the equations given in Figure 1.2. Remember, however, that the angle needs to be defined counterclockwise from the positive x-axis. How can you be sure it is defined properly when working with those equations? Well, remember from algebra that you can divide the Cartesian coordinate plane into four regions:

$$\begin{array}{c|c} \text{II} & \text{I} \\ \hline \text{III} & \text{IV} \end{array}$$
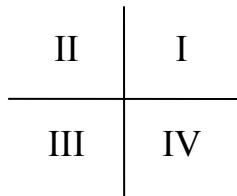
When taking the inverse tangent of a number, the definition of the angle that your calculator gives you depends on which of these regions the vector is in. If the vector is in region I, the angle that your calculator gives you is defined relative to the positive x-axis, just as it should be defined. Thus, if your vector is region I, the angle that your calculator gives you for the inverse tangent function will be defined properly.

However, if the vector is in region II of the Cartesian coordinate plane, then the angle that your calculator gives you is defined relative to the negative x-axis and is negative. In the Cartesian coordinate system, negative angles mean clockwise rotation while positive angles mean counterclockwise rotation. So, when a vector is in region II, your calculator gives you the number of degrees clockwise from the negative x-axis. Thus, if your vector is in region II and your calculator gives you a direction of -60 $^\circ$, this is what it means:

θ = -60°
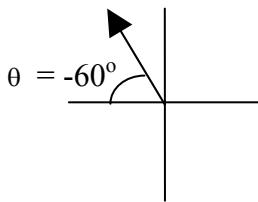
This definition of angle is not proper for our purposes. As a result, we must convert it to the proper definition. It turns out that in both regions II and III of the Cartesian coordinate plane, if you simply add 180 to the angle that your calculator gives you, you will have converted your calculator's answer to an answer in which the angle is defined properly. If the vector is in region IV, then you must add 360 to the calculator's answer in order to get the properly defined angle.

In summary, Figure 1.4 tells you what you must do in order to change your calculator's answer into a properly defined vector angle, based on the region of the Cartesian coordinate system:

FIGURE 1.4
Converting reference angle to vector angle

	do
+ 180.0	nothing
+180.0	+360.0

This figure, of course, does you no good if you can't tell what region of the Cartesian coordinate plane your vector is in. Luckily, however, this is not a difficult task. All you have to do is look at the signs on the vector components. If the x-component and y-component are both positive, then the vector must be in region I. After all, a positive x-component indicates that you are to the right of the origin, while a positive y-component means you are above the origin. The region that is both to the right and above the origin is region I. On the other hand, suppose that both components are negative. Since a negative x-component means left of the origin and a negative y-component means down, you must be in region III, since that is the only region that is to the left and below the origin. See how you do this? Following the same logic, if the x-component is negative and the y-component is positive, you must be in region II. If the x-component is positive and the y-component is negative, however, you must be in region IV.

I want to make sure you understand this by showing you a quick example.

EXAMPLE 1.6

What are the magnitude and direction of vector C in the previous example?

In the previous example, **C** had an x-component of -16.8 miles and a y-component of 12.77 miles. Getting the magnitude is simple, given the equation in Figure 1.2.

$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(-16.8 \text{ miles})^2 + (12.77 \text{ miles})^2} = 21.1 \text{ miles}$$

Getting the angle is a bit more difficult, because it must be defined properly. When we use the equation in Figure 1.2, we get:

$$\theta = \tan^{-1}(\frac{12.77 \text{ miles}}{-16.8 \text{ miles}}) = -37.2^\circ$$

Now that we have an answer, we must figure out the region of the vector. It has a negative x-component (and is therefore left of the origin) and a positive y-component (and is therefore above the x-axis). Thus, it is in region II. According to Figure 1.4, then, we must add 180.0 to it. The properly-defined angle, then, is 142.8°. Thus vector **C** can also be represented as 21.1 miles at an angle of 142.8°.

I went through all of this rather quickly, but it should be review for you. You really should have learned all of this in your first-year course. Thus, if all of this seems a bit "fuzzy" to you, go back and review your first-year physics course.

ON YOUR OWN

1.4 The velocity vector of a car has an x-component of 23 m/sec and a y-component of 11 m/sec. What are the magnitude and direction of the velocity vector?

1.5 Vector **A** has a magnitude of 3.1 m/sec at an angle of 60.0 degrees, and vector **B** has a magnitude of 1.4 m/sec at an angle of 290.0 degrees. What is the sum of these two vectors? Give your answer both graphically and in terms of magnitude and direction.

Unit Vectors

Everything in the previous section should be review for you, and that's why I went through it so quickly. However, I now want to introduce some notation that may be new for you. Since a two-dimensional vector can be represented in terms of its horizontal and vertical components, it is nice to actually define two **unit vectors** – one in the horizontal direction and one in the vertical direction. A unit vector, as shown in the figure below, has a magnitude of 1 and a direction along either the horizontal (x) or vertical (y) axis.

FIGURE 1.5
Two Unit Vectors

Vector **j**:
j = 1, θ = 90⁰
$j_x = 0$, $j_y = 1$

Vector **i**:
i = 1, θ = 0⁰
$i_x = 1$, $i_y = 0$
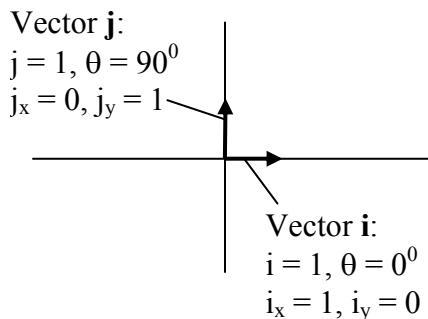
Why do I bother to define these vectors? Think about it. I can define any vector in terms of a horizontal component and a vertical component. Well, suppose I multiply vector **i** by a scalar. What will happen to the magnitude of the vector? Well, since the original magnitude is 1, the new magnitude will be equal to the value of the scalar. What about the direction? Well, unless the scalar is negative, the direction will not change. If the scalar is negative, the direction will simply be reversed. The vector 3·**i**, for example, has a magnitude of 3 and points horizontally to the right. The vector -5·**j**, on the other hand, has a magnitude of 5 and points vertically down.

So what? Think about the fact that we can define a vector by its horizontal and vertical components. Suppose velocity vector **A** has a horizontal component of 5 m/sec ($A_x = 5$ m/sec) and a vertical component of -4 m/sec ($A_y = -4$ m/sec). We could write that in terms of adding unit vectors:

$$\mathbf{A} = (5 \text{ m/sec})\cdot\mathbf{i} \ + \ (\text{-}4 \text{ m/sec})\cdot\mathbf{j}$$

After all, if the x-component is 5 m/sec, then the x-component points to the right and has a magnitude of 5. That's what 5·**i** is. If the y-component is -4 m/sec, then the y-component points down with a magnitude of 4 m/sec. That's what -4·**j** means. In the end, then, we can also note any vector in terms of unit vectors:

$$\text{For any vector } \mathbf{A}, \quad \mathbf{A} = A_x\cdot\mathbf{i} \ + \ A_y\cdot\mathbf{j} \tag{1.1}$$

This is one of the standard ways in which vectors are expressed, so it is important that you understand it.

EXAMPLE 1.7

The acceleration vector of an airplane is 123 m/sec² at θ = 223.2º. Express the vector in terms of unit vectors. Draw a graph of the vector.

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To express a vector in terms of unit vectors, we must have its horizontal and vertical components. Thus, we must convert magnitude and direction into components:

$$A_x = (123 \frac{m}{sec^2}) \cdot \cos(223.2°) = -89.7 \frac{m}{sec^2}$$

$$A_y = (123 \frac{m}{sec^2}) \cdot \sin(223.2°) = -84.2 \frac{m}{sec^2}$$

Now that we have the components, we just multiply **i** by the x-component and **j** by the y-component, and we can express the vector in terms of unit vectors:

$$\mathbf{A} = -89.7 \frac{m}{sec^2} \cdot \mathbf{i} + (-84.2 \frac{m}{sec^2}) \cdot \mathbf{j}$$

Usually, if the vector **j** is multiplied by a negative, we just replace the plus sign with a minus sign, so the more standard answer is:

$$\mathbf{A} = -89.7 \frac{m}{sec^2} \cdot \mathbf{i} - 84.2 \frac{m}{sec^2} \cdot \mathbf{j}$$

 Now remember, the numbers multiplying **i** and **j** are simply the components of the vector. Thus, drawing the vector is a snap:
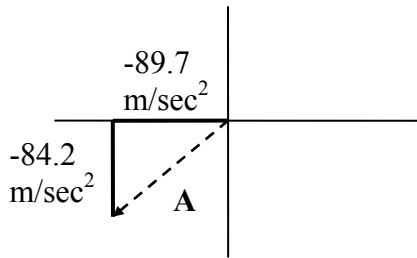
Now remember that to add any two vectors, you simply add the x-components together and the y-components together. Similarly, to subtract vectors, you simply subtract the x-components and the y-components. In this notation, then, vector addition and subtraction is as follows:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x) \cdot \mathbf{i} + (A_y + B_y) \cdot \mathbf{j} \qquad (1.2)$$

$$\mathbf{A} - \mathbf{B} = (A_x - B_x) \cdot \mathbf{i} + (A_y - B_y) \cdot \mathbf{j} \qquad (1.3)$$

We can also express scalar multiplication in this notation. When I multiply a vector by a scalar, I can simply multiply each component by the scalar. Thus, in this notation, scalar multiplication becomes:

$$k{\cdot}\mathbf{A} \;=\; (k{\cdot}A_x){\cdot}\mathbf{i} \;+\; (k{\cdot}A_y){\cdot}\mathbf{j} \qquad \text{(where k is a scalar)} \qquad (1.4)$$

Make sure you understand this new notation by solving the following problems.

ON YOUR OWN

1.6 Draw the vector $\mathbf{A} = (5.0 \text{ m}){\cdot}\mathbf{i} \; - (11 \text{ m}){\cdot}\mathbf{j}$. Give its magnitude and direction.

1.7 What is the sum of vector $\mathbf{A}$ above with vector $\mathbf{B} = (-11 \text{ m}){\cdot}\mathbf{i} \; + \; (22 \text{ m}){\cdot}\mathbf{j}$? What is the difference $\mathbf{A} - \mathbf{B}$? Answer using the notation you learned in this section of the module.

1.8 Write (in unit vector notation) the vector which corresponds to 6 times the vector $\mathbf{C} = (2.0 \text{ m/sec}){\cdot}\mathbf{i} \; + \; (3.0 \text{ m/sec}){\cdot}\mathbf{j}$.

The Dot Product

When we add or subtract two vectors, the result is a vector. When we multiply a vector by a scalar, the result is also a vector. Thus, you might think that when you mathematically manipulate vectors, the result is always a vector. That's not true. One mathematical manipulation is called the **dot product**, and it is important in physics. When you compute the dot product of two vectors, the result is a scalar.

How can two vectors produce a scalar? Well, first let's look at the mechanics of taking the dot product of two vectors, and then I will tell you what the dot product really *means*.

$$\mathbf{A} \bullet \mathbf{B} \;=\; A_x{\cdot}B_x \;+\; A_y{\cdot}B_y \qquad\qquad (1.5)$$

In order to compute the dot product of two vectors, I multiply their x-components together and add that result to the product of their y-components. Notice that there are no vectors on the right hand side of the equation, so the result is, indeed, a scalar.

EXAMPLE 1.8

Given the following vectors:

$\mathbf{A} = \; 14{\cdot}\mathbf{i} \; + \; 11{\cdot}\mathbf{j}$
$\mathbf{B} = \; -2.0{\cdot}\mathbf{i} \; + 3.0{\cdot}\mathbf{j}$

Compute the dot product $\mathbf{A} \bullet \mathbf{B}$.

Using Equation (1.5):

$$\mathbf{A} \bullet \mathbf{B} = A_x \cdot B_x + A_y \cdot B_y$$

$$\mathbf{A} \bullet \mathbf{B} = 14 \cdot (-2.0) + 11 \cdot 3.0 = -28 + 33 = 5$$

The dot product, then, equals 5.

Computing the dot product between two vectors isn't too bad, is it? Of course, being able to perform a mathematical operation is not the same as understanding what that mathematical operation means. Thus, you also need to know the *meaning* behind the dot product. When you take the dot product of two vectors, you are really multiplying the magnitude of the first vector by the component of the second vector which is *parallel* to the first vector. Now I know that sentence is confusing, so I want to try and explain it with a figure.

FIGURE 1.6
The Meaning of the Dot Product
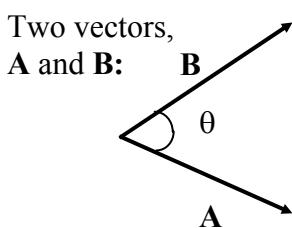
Two vectors, **A** and **B**:

Vector **B** can be split into two perpendicular components: one that is parallel to **A** and one that is perpendicular to **A**:
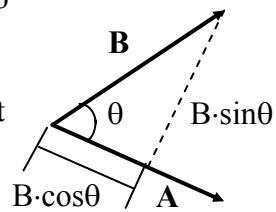
$\mathbf{A} \bullet \mathbf{B} = A[B\cos(\theta)]$

Start by looking at the drawing on the left-hand side of the figure. I have two vectors there, our old friends **A** and **B**. If you put their tails together (remember, moving vectors around is okay as long as you don't change length or direction), you can define θ, the angle between them. Now we already know that any two-dimensional vector can be expressed in terms of two perpendicular components. We usually use the horizontal and vertical components, but that's not necessary. We can really use *any two perpendicular components*. For the purpose of this discussion, then, let's define **B** in terms of a component that is parallel to **A** and one that is perpendicular to **A**. That's what is shown on the right hand side of the figure.

Now look what happens when you split up **B** into those two components. One component lies right on vector **A**. That's the parallel component of vector **B**, and it is calculated by taking the magnitude of the vector (B) and multiplying by the cosine of the angle (θ). Thus, that little section labeled B·cosθ is the component of vector **B** that lies parallel to vector **A**.

When you take the dot product, you are calculating the value of the magnitude of vector **A** multiplied by the length of the *component* of vector **B** which lies parallel to **A**. That's why the last line of text in the figure says that the dot product is A (the magnitude of vector **A**) times Bcosθ (the length of the component of **B** which is parallel to **A**). This is actually another way of expressing the dot product:

$$\mathbf{A} \bullet \mathbf{B} \ = \ \text{A·B·cos}\theta \tag{1.6}$$

The dot product of two vectors, then, can be found one of two ways. If you know the components of the vector, you use Equation (1.5). If you know the magnitudes of each vector and the angle between them, use Equation (1.6). Alternatively, if you know the dot product and magnitudes of the vectors, you can determine the angle between them. See what I mean by studying the following example.

EXAMPLE 1.9

Given the following vectors:

A = 12**i** + 6.0**j**
B = -3.0**i** + 9.0**j**

Determine the angle between the vectors.

To get an idea of just what we are solving for, let's draw the two vectors with each tail at the origin. This is not a necessary step for solving the problem. It simply gives us a visual of the angle.
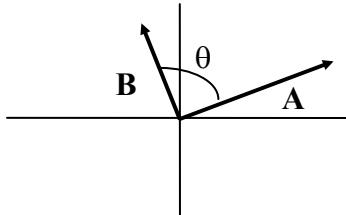
How do we determine the angle? Well notice that we have the components of each vector. Thus, we can determine the magnitude of each. Also, we can determine the dot product using Equation (1.5). Once we have that, then we can solve for the angle using Equation (1.6).

Let's start with the magnitudes of each vector. That's not too hard:

$$A = \sqrt{12^2 + 6.0^2} = 13$$

$$B = \sqrt{-3.0^2 + 9.0^2} = 9.5$$

Now let's figure out the dot product. Since we have the components, we use Equation (1.5):

$$\mathbf{A} \bullet \mathbf{B} = 12 \cdot (-3.0) + 6.0 \cdot 9.0 = 18$$

At this point, we have all of the information in Equation (1.6) except the angle, so we can solve for it:

$$\mathbf{A} \bullet \mathbf{B} = A \cdot B \cdot \cos\theta$$

$$18 = 13 \cdot 9.5 \cdot \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{18}{13 \cdot 9.5}\right) = \cos^{-1}(0.15) = 82^\circ$$

The angle, then, is 82°. Look back at the drawing of the two vectors. Notice that the angle looks very close to a right angle. Thus, the answer we have makes sense based on the drawing.

ON YOUR OWN

1.9 Compute the dot product of the following vectors: $\mathbf{A} = (-2.3 \text{ m}) \cdot \mathbf{i} - (1.2\text{m}) \cdot \mathbf{j}$, $\mathbf{B} = (1.2 \text{ m}) \cdot \mathbf{i} + (4.3 \text{ m}) \cdot \mathbf{j}$.

1.10 Two vectors, $\mathbf{A}$ and $\mathbf{B}$, are defined. The angle between them is 61.0°. Vector $\mathbf{A}$ has a magnitude of 15.1 meters, and the component of $\mathbf{B}$ which is parallel to $\mathbf{A}$ is 1.2 meters long. What is the magnitude of vector $\mathbf{B}$?

The Physical Significance of the Dot Product

In the last section, you learned how to compute the dot product of two vectors and what that means mathematically. However, this is a physics course. It might be nice to learn a new math concept, but this is not a math course. Thus, I would not have taught you about the dot product unless there was some physics behind it.

To learn the physics behind the dot product, you need to recall the concept of **work** from your first-year physics course. Do you remember how we define work mathematically? Try to see if this doesn't jog your memory:

$$W = F_{\parallel} \cdot x \qquad\qquad (1.7)$$

In this equation, "W" represents work , "x" stands for the displacement over which the force was applied, and "$F_\parallel$" is used to indicate the component of the force vector that is parallel to the direction of motion.

To remember why Equation (1.7) uses only that portion of the applied force that is parallel to the motion, consider Figure 1.7.

FIGURE 1.7
Force and Work

Illus. from Arts and Letters Express

In the left side of this figure, a boy is pulling a wagon. If we remove the picture of the boy (right side of the figure) and just look at the vectors involved, we see several things. First, the wagon travels in a straight, horizontal line, as indicated by the horizontal velocity vector (**v**). The force that the boy is applying (**F**) goes in the same direction as the wagon's handle. As a result, his pulling force is not parallel with the motion. Since the force is a two-dimensional vector, we can split it into vertical (F_y) and horizontal (F_x) components. The horizontal component is parallel to the motion and, as you can see, is the only portion of the force that contributes to the motion. In contrast, there is no motion in the perpendicular direction. In other words, the wagon is not moving upwards. In the end, the perpendicular portion of the force fights gravity. Since it is not strong enough to overcome gravity, the wagon does not move up. Remember, when a force is applied but there is no motion, there is no work. Thus, the perpendicular component of the boy's pulling force is wasted. It causes no motion, and therefore it accomplishes no work. That's why only the portion of the force parallel to the motion is considered when calculating the work done by that force.

So, when you are faced with a situation in which you must calculate the work done by a force which takes place over a certain displacement, you can use Equation (1.7). Wait a minute, however. Equation (1.7) tells you to take the magnitude of one vector (the displacement vector, **x**), and multiply it by the magnitude of the component of another vector (the force vector, **F**) which is parallel to the first vector. What does that sound like? It sounds like the *dot product*! Thus, we can re-write the definition of work in dot product notation:

$$W \; = \; \mathbf{F} \bullet \mathbf{x} \qquad\qquad (1.8)$$

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Now remember, work is a scalar. It doesn't tell us anything about direction. That makes sense, since the result of the dot product is a scalar.

If you have been paying close attention, there may be something puzzling you at this point. In the dot product, we take the magnitude of the first vector, and multiply it by the magnitude of the component of the second vector which is parallel to the first vector. Thus, if I were to really write Equation (1.7) in dot product form, it should read W=**x•F**. However, look at Equation (1.6). This equation tells us that the dot product is commutative. After all,

$$\mathbf{A}\bullet\mathbf{B} \; = \; A{\cdot}B{\cdot}\cos\theta$$

$$\mathbf{B}\bullet\mathbf{A} \; = \; B{\cdot}A{\cdot}\cos\theta$$

Notice that $A{\cdot}B{\cdot}\cos\theta$ and $B{\cdot}A{\cdot}\cos\theta$ are equivalent. Thus, **A•B** and **B•A** are equivalent. Thus, the order in which you take the dot product does not matter.

The dot product is commutative: A•B is the same as B•A

This is important and worth remembering.

Now that you know the physical significance of the dot product, solve the following "on your own" problems.

ON YOUR OWN

1.11 A particle undergoes a displacement **x** = (1.5 m)·**i** - (2.3 m)·**j** while being acted upon by a constant force **F** = (5.6 N)·**i** - (3.4 N)·**j**. What is the work done?

1.12 A person applies a force of 16.6 Newtons to an object as the object travels 9.2 meters. If the work done was 14.5 J, what was the angle between the force vector and the displacement vector?

The Cross Product

The dot product of two vectors produces a scalar. It only makes sense that if there is a way to multiply two vectors to produce a scalar, there must be a way to multiply two vectors to produce a vector. Indeed, there is. We call it the **cross product**. In the cross product, we are still multiplying the magnitude of one vector with the magnitude of another vector. In this case, however, we are multiplying the magnitude of the first vector with the component of the second vector which is *perpendicular* to the first vector. In addition to using a different component of the second vector, the cross product also produces a *vector, not a scalar*. Thus, the cross product has both magnitude and direction. Let's deal with the magnitude first.

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FIGURE 1.8
The Magnitude of the Cross Product
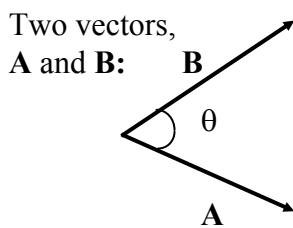
Two vectors, **A** and **B:**

Vector **B** can be split into two perpendicular components: one that is parallel to **A** and one that is perpendicular to **A:**
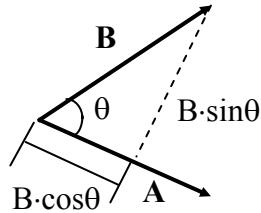
$|A \times B| = A[B\sin(\theta)]$

Once again, we have our old friends vector **A** and vector **B**. They can be drawn tail-to-tail, and the resulting angle between them is called θ. If we split vector **B** into two components: one parallel to **A** and one perpendicular to **A**, we find that the perpendicular component is B·sinθ. Thus, since the cross product involves multiplying the magnitude of the first vector and the magnitude of the component of the second vector which is perpendicular to the first, the magnitude of the cross product is:

$$|A \times B| = A \cdot B \cdot \sin\theta \qquad (1.9)$$

The vertical lines enclosing **A** x **B** simply mean "magnitude." Thus, "|**A** x **B**|" means "the magnitude of the vector **A** x **B**."

So that's how we get the magnitude of the cross product. The cross product produces a vector, however, so there is also direction to consider. How do we come up with the direction of the cross product? We use something called the **right hand rule**.

Right hand rule - To determine the direction of the cross product **A** x **B**, take your right hand and point your fingers in the direction of **A**. Then, curl your fingers towards **B**, along the arc of the angle between the vectors. Your thumb will then point in the direction of the cross product

The right hand rule is illustrated in Figure 1.9.

FIGURE 1.9
The Right hand Rule

Illus. By Megan Whitaker

direction of the cross product
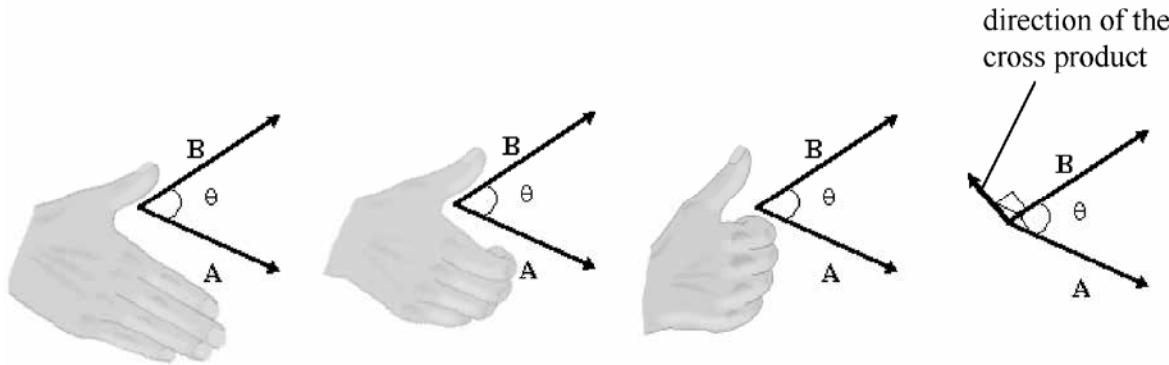
Point the fingers of your right hand in the direction of the first vector.

Curl your fingers along the arc of the angle between the vectors.

Your thumb points in the direction of the cross product.

In this example, the thumb points up out of the page. That's the direction of the cross product.

When determining the direction of the cross product, then, you take your right hand and point your fingers in the direction of the first vector. Then, you curl your fingers towards the second vector, along the arc of the angle between them. Your thumb will then point in the direction of the cross product. Think about what this means for a moment.

The direction of the cross product of two vectors will <u>always</u> be perpendicular to *both* vectors.

The right hand rule has another implication. Look at Figure 1.9 and use the right hand rule to determine the direction of **B** x **A**. Remember, you point the fingers on your right hand in the direction of the *first vector* (**B**), and you then curl towards the *second vector* (**A**). Where does your thumb point? It points down towards the paper. That's the *opposite direction* as that shown in the figure. Thus, unlike the dot production, the cross product is *not commutative*.

The cross product is not commutative: A x B = -(B x A)

This is important to remember.

Okay, there is one more thing you need to learn about the cross product. You need to know how to calculate the cross product using unit vector notation. However, before you can do that, you need to see how we define unit vectors in three-dimensional space.

FIGURE 1.10
Three-Dimensional Space in Terms of Unit Vectors
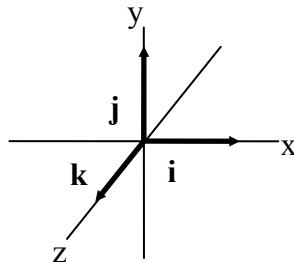
When we add another dimension (the z-axis), we can represent that in unit vector notation with just another unit vector. Notice that **i** is still the unit vector in the horizontal direction and **j** is still the unit vector in the vertical direction. To represent three-dimensional space, then, we simply add a third unit vector, **k**, which points out of the plane of the paper, towards you. If we multiply by -1, the unit vector **-k** points behind the paper.

Now that you know how three-dimensional space is represented in vector notation, you can learn how we compute the cross product. First, let's start with the simple case of two-dimensional vectors.

For two-dimensional vectors in the i/j plane: A x B = $(A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$ (1.10)

Notice the restriction placed on this equation. To use this equation, you must be taking the cross product of two-dimensional vectors which exist only in the plane defined by the horizontal (**i**) and vertical (**j**) axes. Thus, this is a pretty restrictive equation. However, most of the physics problems that you do will involve such vectors, so it is probably what you will use most often for calculating cross products.

For completeness sake, I will give you the total equation for calculating the cross product between any three-dimensional vectors:

$$\mathbf{A} \times \mathbf{B} = (A_y \cdot B_z - A_z \cdot B_y) \cdot \mathbf{i} + (A_z \cdot B_x - A_x \cdot B_z) \cdot \mathbf{j} + (A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$$ (1.11)

Notice that Equation (1.11) reduces to Equation (1.10) for two-dimensional vectors in the **i/j** plane. After all, the z-component of such a two-dimensional vector is zero. Thus, the term $(A_y \cdot B_z - A_z \cdot B_y) \cdot \mathbf{i}$ is zero, as is the term $(A_z \cdot B_x - A_x \cdot B_z) \cdot \mathbf{j}$. As a result, the only term in the equation that is non-zero is $(A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$, and that gives us Equation (1.10).

I know that this is a lot to throw at you, but hopefully the following example problems will clear up any confusion that may still be in your mind.

EXAMPLE 1.10

A velocity vector has a magnitude of 56.1 m/sec and an angle of 45.0 degrees. Another velocity vector has a magnitude of 12.2 m/sec and an angle of 290.1 degrees. Calculate the magnitude of the cross product and give the vector in unit vector notation.

  Calculating the magnitude of the cross product is not bad at all. We just use Equation (1.9). To do that, however, we need the angle between the vectors. To determine that we will have to draw the two vectors:
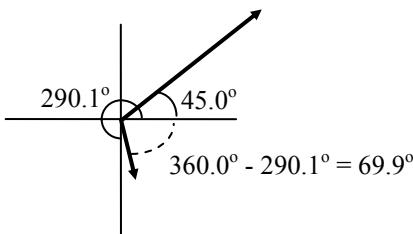
If the angle of the second vector as defined from the positive x-axis is 290.1, then the angle from the positive x-axis *down* to the vector must be $69.9°$, because the total angle must be $360.0°$. Well, the angle from the positive x-axis *up* to the first vector is $45.0°$. The angle between the two vectors, then, must be $69.9° + 45.0° = 114.9°$. Now we can use Equation (1.9):

$$|\textbf{A} \times \textbf{B}| = A \cdot B \cdot \sin\theta = (56.1 \text{ m/sec}) \cdot (12.2 \text{ m/sec}) \cdot \sin(114.9°) = 621 \text{ m}^2/\text{sec}^2$$

The magnitude of the cross product, then, is <u>$621 \text{ m}^2/\text{sec}^2$</u>.

  What about the direction? For that, we use the right hand rule. We point the fingers of our right hand in the direction of the first vector, then we curl along the arc of the $114.9°$ angle in between the vectors. When we do that, our thumb points down into the page. Thus, the vector is in the negative **k** direction. That tells us what we need to know for unit vector notation. After all, we know the vector's magnitude ($621 \text{ m}^2/\text{sec}^2$), and we know that it is pointed in the negative **k** direction. Thus, the vector is <u>$-(621 \text{ m}^2/\text{sec}^2) \cdot \textbf{k}$</u>.

Given the following vectors, calculate A x B and determine the angle between the vectors:

$$\textbf{A} = 3.4 \cdot \textbf{i} + 4.5 \cdot \textbf{j}$$
$$\textbf{B} = 2.4 \cdot \textbf{i} + 1.1 \cdot \textbf{j}$$

  Notice that these two vectors are two-dimensional and have only **i** and **j** unit vectors. Thus, they are in the **i**/**j** plane, and we can use the simpler version of the cross product equation, Equation (1.10):

$$\mathbf{A} \times \mathbf{B} = (A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = [3.4 \cdot 1.1 - 4.5 \cdot (2.4)] \cdot \mathbf{k} = \underline{-7 \cdot \mathbf{k}}$$

That's the cross product. Based solely on the unit vector notation, you know that it is pointing behind the plane of the paper. What about the angle? Well, we now know the magnitude of the vector (if the vector is composed solely of 7 times $\mathbf{k}$, then the magnitude is 7, because $\mathbf{k}$ has a magnitude of 1), so we can use Equation (1.9):

$$|\mathbf{A} \times \mathbf{B}| = A \cdot B \cdot \sin\theta$$

Remember, "$|\mathbf{A} \times \mathbf{B}|$" means "magnitude of $\mathbf{A} \times \mathbf{B}$," so that's 7. We don't have the magnitudes of $\mathbf{A}$ and $\mathbf{B}$, but we can use their components to calculate them:

$$A = \sqrt{3.4^2 + 4.5^2} = 5.6$$
$$B = \sqrt{(2.4)^2 + 1.1^2} = 2.6$$

Now we can use Equation (1.9):

$$|\mathrm{A x B}| = A \cdot B \cdot \sin\theta$$

$$7 = (5.6) \cdot (2.6) \cdot \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{7}{(5.6) \cdot (2.6)}\right) = \underline{30°}$$

Given the following vectors, calculate the cross product and the angle between them:

$$\mathbf{A} = -1.4 \cdot \mathbf{i} + 4.2 \cdot \mathbf{j} - 5.6 \cdot \mathbf{k}$$
$$\mathbf{B} = 2.4 \cdot \mathbf{i} - 1.1 \cdot \mathbf{j} + 3.2 \cdot \mathbf{k}$$

This is essentially the same as the problem above. However, these are three-dimensional vectors, so we have to use the larger formula, Equation (1.11):

$$\mathbf{A} \times \mathbf{B} = (A_y \cdot B_z - A_z \cdot B_y) \cdot \mathbf{i} + (A_z \cdot B_x - A_x \cdot B_z) \cdot \mathbf{j} + (A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = (4.2 \cdot 3.2 - [-5.6] \cdot [-1.1]) \cdot \mathbf{i} + (-5.6 \cdot 2.4 - [-1.4] \cdot 3.2) \cdot \mathbf{j} + ([-1.4] \cdot [-1.1] - 4.2 \cdot 2.4) \cdot \mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = \underline{7 \cdot \mathbf{i} - 9 \cdot \mathbf{j} - 9 \cdot \mathbf{k}}$$

Just as we did before, we can calculate the magnitudes of the two vectors and then determine the angle between them using Equation (1.9). The only difference now is that the magnitude of the cross product is not as easy to determine. We must calculate it like we calculate the magnitudes of all vectors:

$$A = \sqrt{(-1.4)^2 + 4.2^2 + (-5.6)^2} = 7.1$$

$$B = \sqrt{2.4^2 + 1.1^2 + 3.2^2} = 4.1$$

$$|\,AxB\,| = \sqrt{7^2 + (-9)^2 + (-9)^2} = 15$$

Now that we have all of the magnitudes involved, we can use Equation (1.9):

$$|\,AxB\,| = A \cdot B \cdot \sin\theta$$

$$15 = (7.1) \cdot (4.1) \cdot \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{15}{(7.1) \cdot (4.1)}\right) = \underline{31^o}$$

ON YOUR OWN

1.13 Vector **A** is defined as 3.2 feet at 45.0°, and vector **B** is defined as 1.1 feet at 70.1°. What is the cross product? Give your answer in unit vector notation.

1.14 Given the following vectors, calculate the cross product and the angle between them:

$$\mathbf{A} = \text{-7.1·}\mathbf{i} + 4.2·\mathbf{j}$$
$$\mathbf{B} = 3.4·\mathbf{i} - 4.1·\mathbf{j}$$

The Physical Significance of the Cross Product

As was the case with the dot product, there is physical significance to the cross product. We will apply the cross product in at least three different areas of physics, but for right now, I will concentrate on only one: the concept of **torque**. As you learned in your first-year physics course, when we apply a force some distance away from an axis of rotation, the result is a torque, which can cause rotational motion. In your first-year physics course, you learned the equation used to calculate torque.

$$\tau = F_\perp \cdot r \tag{1.12}$$

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Where "τ" represents torque, "r" represents the magnitude of the vector drawn from the axis of rotation to the point at which the force is applied, and "$F_\perp$" represents the component of the force which is perpendicular to that vector. Figure 1.11 illustrates the concept of torque.

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FIGURE 1.11
Torque
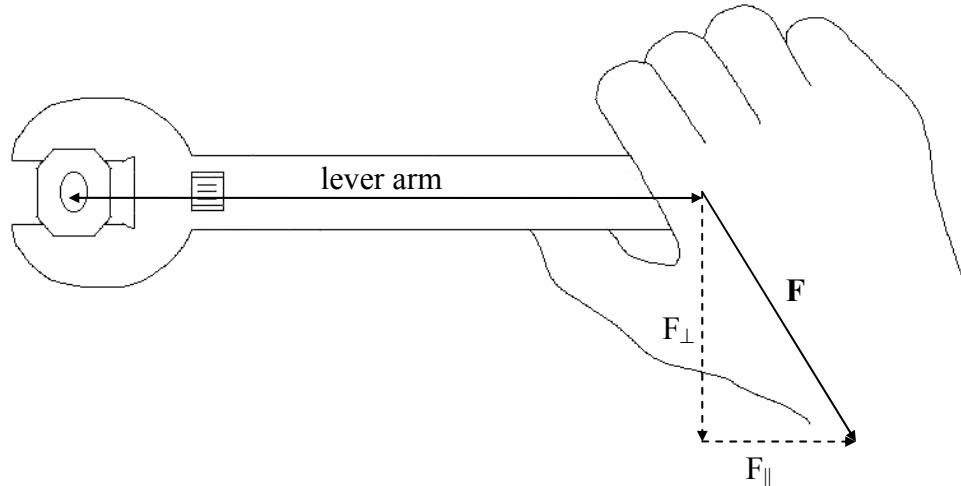
Now remember what torque is. It is the counterpart of force when one is considering *rotational motion*. Remember, force causes acceleration in a straight line. Torque causes rotational acceleration. In the figure, the wrench is going to turn the screw. To do that, it will have to give the screw rotational acceleration so that it starts to turn in a circle. Torque is the impetus which will cause that rotational acceleration.

Notice from the figure that only a portion of the force used can generate torque. Any component of the force that is parallel to the vector defined from the axis of rotation to the point at which the force is applied (the **lever arm**) is lost. Thus, the magnitude of the torque is given by the magnitude of the lever arm times the component of the force which is perpendicular to the lever arm.

Well, since the cross product takes the magnitude of a vector and multiplies it by the magnitude of a second vector's component which is perpendicular to the first vector, torque can be calculated using the cross product.

$$\tau = r \text{ x } F \tag{1.13}$$

There are two things to note about this equation. First, torque is a vector. Remember, the cross product results in a vector. That's why the "τ" is bold. Second, remember that the cross product

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is NOT commutative. Thus, the order of the vectors *is* important, so one must take the vector of the lever arm (**r**) and cross it with the vector representing the force (**F**), *in that order*.

Since you already know how to do cross products, calculating the torque is pretty easy. Thus, I will not give you any examples of it. However, you should perform the following "on your own" problems to make sure you understand how to use the cross product to calculate torque.

ON YOUR OWN

1.15 A person applies a force **F** = (15 N)·**i** + (23 N)·**j** on a lever arm **r** = (1.2 m)·**i** + (1.1 m)·**j**. What is the vector that represents the torque?

1.16 A man is trying to turn a bolt. He exerts a force of 15.2 N with a pipe that creates a lever arm which is 0.36 m long. If the plumber succeeds in producing 4.9 N·m of torque, what is the angle between the force he is exerting and the wrench? Given the diagram below, is the torque pointing up above the plane of the paper or back behind the plane of the paper?

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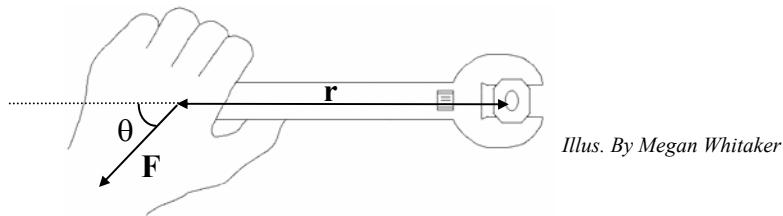
Summing Up

This module contained some concepts which were review for you and others which were new. That will be the case with most of the modules in this course. In each module, I will review some of the highlights of your first-year physics course and then go deeper into each subject. In addition, completely new concepts will be brought in from time to time. Thus, if there is something that really baffles you in this course, you should review your first-year course to see if it is explained there.

ANSWERS TO THE ON YOUR OWN PROBLEMS

1.1 To solve this problem, we simply have to convert miles to centimeters and hours to seconds. It doesn't matter that they are both a part of our original unit. As long as we use the factor-label method, everything will work out:

$$\frac{65 \text{ miles}}{1 \text{ hour}} \times \frac{1609 \text{ m}}{1 \text{ mile}} \times \frac{1 \text{cm}}{0.01 \text{m}} \times \frac{1 \text{ hour}}{3600 \text{ s}} = 2900 \frac{\text{cm}}{\text{s}}$$

Notice that I converted from miles to meters and then meters to centimeters. I had to do that, since I was not given a direct relationship between miles and centimeters. Sixty-five miles per hour is the same as 2900 cm/s.

1.2 We can't use the conversion of 3 feet in 1 yard right away, because we want to convert from square feet into square yards. Thus, we need a relationship between those quantities. We can get such a relationship by squaring both sides of the relationship that we do have:

$$3 \text{ ft} = 1 \text{ yd}$$

$$9 \text{ ft}^2 = 1 \text{ yd}^2$$

Now we can do the conversion:

$$\frac{1600 \text{ ft}^2}{1} \times \frac{1 \text{ yd}^2}{9 \text{ ft}^2} = 180 \text{ yd}^2$$

The house has an area of 180 square yards. If your answer was 177.8, go back and review significant figures from your first-year course.

1.3 To convert from Joules ($\frac{\text{kg} \cdot \text{m}^2}{\text{sec}^2}$) to ergs ($\frac{\text{g} \cdot \text{cm}^2}{\text{sec}^2}$), I need to convert kg into g and m^2 into cm^2. That's not bad. We know that 1 kg = 1,000 g. We also know:

$$1 \text{ cm} = 0.01 \text{ m}$$

To get the relationship between cm^2 and m^2, we just square both sides:

$$1 \text{ cm}^2 = 0.0001 \text{ m}^2$$

This leads us to:

$$\frac{151 \text{ kg} \cdot \text{m}^2}{1 \text{ sec}^2} \times \frac{1,000 \text{ g}}{1 \text{ kg}} \times \frac{1 \text{ cm}^2}{0.0001 \text{ m}^2} = 1.51 \times 10^9 \frac{\text{g} \cdot \text{cm}^2}{\text{sec}^2}$$

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1.4 In this problem, we are given the x- and y-components of a vector and are asked to calculate its magnitude and direction. Getting the magnitude is not too bad:

$$\text{Magnitude} = \sqrt{V_x^{\,2} + V_y^{\,2}} = \sqrt{(23\,\frac{m}{sec})^2 + (11\,\frac{m}{sec})^2} = 25\,\frac{m}{sec}$$

To get the angle, we start with this equation:

$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{11\,\frac{m}{sec}}{23\,\frac{m}{sec}}\right) = 26^{\circ}$$

We aren't necessarily finished yet, however. We have to determine which region of the Cartesian coordinate system that the vector is in. Since both its components are positive, this tells us that the vector is to the right and above the origin, which means that the vector is in region I. According to our rules, we don't need to do anything to the result of the equation when the vector is in region I, so 26° is the proper angle. Thus, <u>the vector has magnitude of 25 m/sec and direction of 26°</u>.

1.5   In this problem, we are asked to add two-dimensional vectors together. Before we do this mathematically, let's do it graphically:
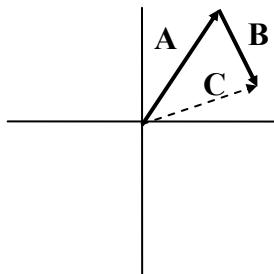
As we learned before, the dotted arrow (vector **C**) gives us the sum.

The first step in adding vectors mathematically is to break both vectors down into their components:

$$A_x = (3.1\,\frac{m}{sec})\cdot\cos(60.0^{\circ}) = 1.6\,\frac{m}{sec}$$

$$A_y = (3.1\,\frac{m}{sec})\cdot\sin(60.0^{\circ}) = 2.7\,\frac{m}{sec}$$

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$$B_x = (1.4 \, \frac{m}{sec}) \cdot \cos(290.0°) = 0.48 \, \frac{m}{sec}$$

$$B_y = (1.4 \, \frac{m}{sec}) \cdot \sin(290.0°) = -1.3 \, \frac{m}{sec}$$

Now that we have the individual components, we can add them together.

$$C_x = A_x + B_x = 1.6 \, \frac{m}{sec} + 0.48 \, \frac{m}{sec} = 2.1 \, \frac{m}{sec}$$

$$C_y = A_y + B_y = 2.7 \, \frac{m}{sec} + -1.3 \, \frac{m}{sec} = 1.4 \, \frac{m}{sec}$$

Now that we have the components to our answer, we can get the magnitude and direction of the sum.

$$\text{Magnitude} = \sqrt{C_x{}^2 + C_y{}^2} = \sqrt{(2.1 \, \frac{m}{sec})^2 + (1.4 \, \frac{m}{sec})^2} = 2.5 \, \frac{m}{sec}$$

$$\theta = \tan^{-1}(\frac{C_y}{C_x}) = \tan^{-1}(\frac{1.4 \, \frac{m}{sec}}{2.1 \, \frac{m}{sec}}) = 34°$$

Since the x- and y-components are both positive, the vector is in the first region of the Cartesian coordinate plane. This is consistent with the graphical answer we drew to begin with, and it means that we do not need to do anything to the result the equation. Thus, the sum of vectors **A** and **B** has a magnitude of 2.5 m/sec at a direction of 34°.

1.6 To draw the vector, we simply need to realize that the number multiplying **i** is the x-component and the number multiplying **j** is the y-component.
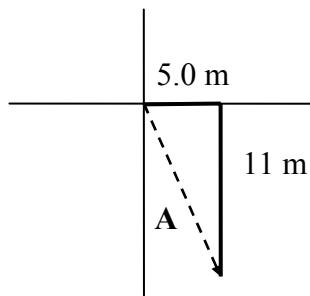
To get the magnitude and direction, we just use the formulas in Figure 1.2.

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{(5.0\,\text{m})^2 + (-11\,\text{m})^2} = 12\ \text{m}$$

$$\theta = \tan^{-1}\left(\frac{-11\ \text{m}}{5.0\ \text{m}}\right) = -66°$$

Now remember, the angle may not be defined correctly at this point. Based on the components, we see that this vector is in region IV of the Cartesian coordinate plane. Thus, we must add $360.0°$ to it. Thus, the answer is <u>12 m at a direction of 294°</u>.

1.7 To add vectors, we add the x-components and y-components. In unit vector notation, the formula is given by Equation (1.2):

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} = (5\ \text{m} + -11\ \text{m})\mathbf{i} + (-11\ \text{m} + 22\text{m})\mathbf{j} = \underline{(-6\ \text{m})\mathbf{i} + (11\ \text{m})\mathbf{j}}$$

The difference is given by Equation (1.3):

$$\mathbf{A} - \mathbf{B} = (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} = (5\ \text{m} - -11\text{m})\mathbf{i} + (-11\ \text{m} - 22\ \text{m})\mathbf{j} = \underline{(16\ \text{m})\mathbf{i} - (33\ \text{m})\mathbf{j}}$$

1.8 In unit vector notation, scalar multiplication is given by Equation (1.2):

$$k\mathbf{A} = (kA_x)\mathbf{i} + (kA_y)\mathbf{j}$$

$$6\mathbf{C} = (6\times2.0\ \text{m/sec})\mathbf{i} + (6\times3.0\ \text{m/sec})\mathbf{j} = \underline{(12\ \text{m/sec})\mathbf{i} + (18\ \text{m/sec})\mathbf{j}}$$

1.9 Using Equation (1.5):

$$\mathbf{A} \bullet \mathbf{B} = A_x{\cdot}B_x + A_y{\cdot}B_y$$

$$\mathbf{A} \bullet \mathbf{B} = (-2.3\ \text{m}){\cdot}(1.2\ \text{m}) + (-1.2\ \text{m}){\cdot}(4.3\ \text{m}) = -8.0\ \text{m}^2$$

The dot product, then, equals <u>-8.0 m²</u>. What does the minus sign mean? Well, remember, the dot product is the product of the magnitude of **A** and the magnitude of the component of **B** parallel to **A**. If that product is negative, it means the component of **B** parallel to **A** is negative. That means the component of **B** parallel to **A** points opposite of **A**. Draw the two vectors to see what I mean:

Component of **B** parallel to **A** points in the opposite direction compared to **A**. Thus, compared to **A**, it is negative. This leads to a negative dot product.
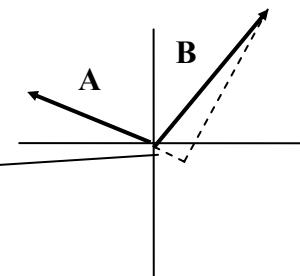
1.10 This problem requires you to understand that the dot product **A**•**B** is the magnitude of **A** times the magnitude of the component of **B** which lies parallel to **A**. You are given those two things, so you can compute the dot product:

$$\mathbf{A} \bullet \mathbf{B} = (\text{magnitude of } \mathbf{A}) \cdot (\text{magnitude of the component of } \mathbf{B} \text{ parallel to } \mathbf{A})$$

$$\mathbf{A} \bullet \mathbf{B} = (15.1 \text{ m}) \cdot (1.2 \text{ m}) = 18 \text{ m}^2$$

Now that we have the dot product, we can solve Equation (1.6) for the magnitude of **B**.

$$\mathbf{A} \bullet \mathbf{B} = A \cdot B \cdot \cos\theta$$

$$18 \text{ m}^2 = (15.1 \text{ m}) \cdot B \cdot \cos(61.0°)$$

$$B = \frac{18 \text{ m}^2}{(15.1 \text{ m}) \cdot \cos(61.0°)} = \underline{2.5 \text{ m}}$$

1.11 This is a straightforward application of Equation (1.8):

$$W = \mathbf{F} \bullet \mathbf{x} = (5.6 \text{ N}) \cdot (1.5 \text{ m}) + (3.4 \text{ N}) \cdot (2.3 \text{ m}) = 16.2 \text{ N·m} = \underline{16.2 \text{ J}}$$

Remember from your first-year course that a N·m is the standard unit for energy or work, and that unit is called the **Joule** in honor of James Prescott Joule.

1.12 Remember, work is the dot product of **F** and **x**. Thus,

$$W = F \cdot x \cdot \cos\theta$$

In this problem, we have all of the variables except θ, so we can solve for it:

$$W = F \cdot x \cdot \cos\theta$$

$$14.5 \text{ J} = (16.6 \text{ N}) \cdot (9.2 \text{ m}) \cdot \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{14.5 \text{ N} \cdot \text{m}}{(16.6 \text{ N}) \cdot (9.2 \text{ m})}\right) = \underline{85°}$$

Notice that in the last line, I replaced the unit Joule with its definition, N·m, to show that the units cancel out. Remember, when dealing with a trigonometric function, the argument should have no units. Thus, any units in the problem need to fully cancel, as they do above.

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1.13   To calculate the magnitude of the cross product, we just use Equation (1.9). To do that, however, we need the angle between the vectors:
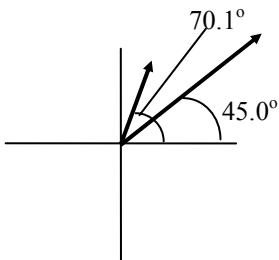
This angle is easy, it's just 70.1° - 45.0° = 25.1°.   Now we can use Equation (1.9):

$$|\mathbf{A} \times \mathbf{B}| = A \cdot B \cdot \sin\theta = (3.2 \text{ feet}) \cdot (1.1 \text{ feet}) \cdot \sin(25.1°) = 1.5 \text{ feet}^2$$

To determine direction, we use the right hand rule. We point the fingers of our right hand in the direction of the first vector, then we curl along the arc of the 25.1° angle in between the vectors. When we do that, our thumb points up out of the page. Thus, the vector is in the positive **k** direction. That tells us what we need to know for unit vector notation. Thus, the vector is (1.5 feet²)·**k**.

1.14 Notice that the vectors are two-dimensional and have only **i** and **j** unit vectors. Thus, we can use the simpler version of the cross product formula, Equation (1.10):

$$\mathbf{A} \times \mathbf{B} = (A_x \cdot B_y - A_y \cdot B_x) \cdot \mathbf{k}$$

$$\mathbf{A} \times \mathbf{B} = [(-7.1) \cdot (-4.1) - 4.2 \cdot 3.4] \cdot \mathbf{k} = \underline{15 \cdot \mathbf{k}}$$

That's the cross product. What about the angle? Well, we just need the magnitudes of all three vectors and Equation (1.9). The magnitude of the cross product is easy; it's just 15. What about **A** and **B**?

$$A = \sqrt{(-7.1)^2 + (4.2)^2} = 8.2$$
$$B = \sqrt{(3.4)^2 + (-4.1)^2} = 5.3$$

Now we can use Equation (1.9):

$$|\mathbf{A} \times \mathbf{B}| = A \cdot B \cdot \sin\theta$$

$$15 = (8.2) \cdot (5.3) \cdot \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{15}{(8.2) \cdot (5.3)}\right) = \underline{2.0 \times 10^{1\; \circ}}$$

Notice that I had to put the answer in scientific notation. The rules indicate that I must have 2 significant figures. The number 20 has only 1 significant figure. Thus, I had to make that zero significant by putting the whole number in scientific notation.

1.15   This is an easy one. We are given the vectors for **r** and **F**, so we can just use Equation (1.13). Since these are two-dimensional vectors in the **i**/**j** plane, we can use the simplified cross product formula, Equation (1.10).

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (r_x \cdot F_y - r_y \cdot F_x) \cdot \mathbf{k} = [(1.2\text{ m}) \cdot (23\text{ N}) - (1.1\text{ m}) \cdot (15\text{ N})] \cdot \mathbf{k} = \underline{(11\text{ N·m}) \cdot \mathbf{k}}$$

1.16   In this problem, we are dealing only with magnitudes. Since we want to figure out the angle between the vectors, we are going to use Equation (1.9):

$$|\mathbf{A} \times \mathbf{B}| = A \cdot B \cdot \sin\theta$$

$$4.9\text{ N} \cdot \text{m} = (0.36\text{ m}) \cdot (15.2\text{ N}) \cdot \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{4.9\text{ N} \cdot \text{m}}{(0.36\text{ m}) \cdot (15.2\text{ N})}\right) = \underline{64°}$$

To determine the direction, we use the right hand rule. Point the fingers of your right hand from the nut to the hand. That is along the lever arm. Now, curl your fingers towards the force, along the arc of the angle between the lever arm and the force. Your thumb should be pointed up away from the paper. Thus, the torque points above the plane of the paper.

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